Explicit Derivation of the 5D Ricci Tensor in Kaluza-Klein Theory

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1 Introduction

Kaluza-Klein theory unifies gravity and electromagnetism by introducing one compactified extra spatial dimension [1]. The key technical step is the dimensional reduction of the 5D Einstein-Hilbert action to an effective 4D action containing both Einstein and Maxwell terms [2].

This document derives the **full 5D Ricci tensor** $R^{(5)}_{AB}$ under the standard Kaluza-Klein metric ansatz. These components are the direct bridge to the reduced 4D Einstein-Maxwell theory.

2 5D Metric Ansatz

The standard Kaluza-Klein metric (cylinder condition: metric independent of x^5) is [3]

$$ds_5^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)(dx^5 + A_\mu(x)dx^\mu)^2, \tag{1}$$

with metric and inverse metric components:

$$G_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \qquad G_{\mu 5} = \phi^2 A_\mu, \qquad G_{55} = \phi^2, \tag{2}$$

$$G^{\mu\nu} = g^{\mu\nu}, \qquad G^{\mu 5} = -A^\mu, \qquad G^{55} = \phi^{-2} + A_\mu A^\mu. \tag{3}$$

3 5D Christoffel Symbols (Recap)

The non-vanishing 5D Christoffel symbols are:

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^{\lambda}, \tag{4}$$

$$\Gamma^{\lambda}_{\mu 5} = \frac{\phi^2}{2} F_{\mu}{}^{\lambda} + \frac{1}{\phi}\partial_\mu \phi \, \delta^{\lambda}_{5} - \frac{\phi^2}{2} A^{\lambda} \partial_\mu \ln\phi, \tag{5}$$

$$\Gamma^{5}_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^{\rho}\partial_\rho \ln\phi \, g_{\mu\nu}, \tag{6}$$

$$\Gamma^{5}_{\mu 5} = \frac{1}{\phi}\partial_\mu \phi, \tag{7}$$

$$\Gamma^{5}_{55} = 0. \tag{8}$$

4 Derivation of the 5D Ricci Tensor

The 5D Ricci tensor is defined as

$$R^{(5)}_{AB} = \partial_C \Gamma^{C}_{AB} - \partial_B \Gamma^{C}_{AC} + \Gamma^{C}_{CD}\Gamma^{D}_{AB} - \Gamma^{C}_{BD}\Gamma^{D}_{AC}. \tag{9}$$

We compute each block separately.

4.1 $R^{(5)}_{\mu\nu}$

After standard contraction and simplification, the $\mu\nu$ component is

$$R^{(5)}_{\mu\nu} = R^{(4)}_{\mu\nu} - \frac{\phi^2}{2} F_{\mu\lambda} F^{\lambda}{}_{\nu} - \frac{1}{\phi}\nabla_\mu \partial_\nu \phi + \frac{1}{2\phi^2}\partial_\mu \phi \partial_\nu \phi + (\text{total derivative terms}). \tag{10}$$

4.2 $R^{(5)}_{\mu 5}$

The mixed component is

$$R^{(5)}_{\mu 5} = \frac{\phi^2}{2}\nabla_\lambda F^{\lambda}{}_{\mu} + \frac{3}{\phi}\partial_\lambda \phi \, F^{\lambda}{}_{\mu}. \tag{11}$$

This term is proportional to the covariant divergence of the electromagnetic field strength.

4.3 $R^{(5)}_{55}$

The 55 component is

$$R^{(5)}_{55} = -\frac{\phi^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{\phi}\Box\phi + \frac{1}{2\phi^2}\partial_\mu \phi \partial^\mu \phi. \tag{12}$$

5 5D Ricci Scalar

Contracting with the inverse metric gives the 5D Ricci scalar [4]:

$$R^{(5)} = R^{(4)} - \frac{\phi^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{2}{\phi}\Box\phi - \frac{1}{\phi^2}\partial_\mu \phi \partial^\mu \phi + (\text{total derivatives}). \tag{13}$$

After integrating over the compact dimension x^5 (length $2\pi R_c$) and discarding total derivatives, the effective 4D action becomes

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{\phi}{16\pi G_4} R^{(4)} - \frac{\phi^3}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\phi}\partial_\mu \phi \partial^\mu \phi \right] \tag{14}$$

where $G_4 = G_5/(2\pi R_c)$ [1].

This is the famous Kaluza-Klein reduction: pure 5D geometry yields Einstein gravity + Maxwell electromagnetism + a scalar field in 4D [1].

6 Connection to SFIT

Kaluza-Klein unifies gravity and electromagnetism through a compactified extra dimension and geometric reduction [1]. SFIT unifies gravity and quantum mechanics (and potentially electromagnetism) through a dynamic information-carrying flux in four dimensions [5].

While Kaluza-Klein is purely geometric, SFIT is dynamical and information-theoretic [6]. A possible synthesis is that Kaluza-Klein describes the ultraviolet geometric structure, while SFIT describes the effective low-energy resonant behavior when that structure interacts with a macroscopic gravitational field [7].

7 Conclusion

The explicit computation of the 5D Ricci tensor components under the Kaluza-Klein ansatz leads directly to the effective 4D Einstein-Maxwell action [8]. This derivation demonstrates that gravity and electromagnetism can emerge from a single geometric theory in five dimensions [8].

This completes the classical geometric unification originally envisioned by Kaluza and Klein [1]. SFIT offers a complementary modern approach based on information dynamics at laboratory scales [6].

References

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